PRIOR DISTRIBUTIONS IN DYNARE

(TO BE COMPLETED)

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1. Gamma Distributions

Definition 1.1. *The Gamma function is defined as follows:*

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}\mathrm{d}x$$

for any $x > 0$, zero elsewhere.

One can easily prove that the following identities hold: $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)\Gamma(n-1)$.

Definition 1.2. *A positive real random variable has a gamma distribution with parameters $\alpha > 0$ (shape) and $\beta > 0$ (scale) iff its probability density function is given by the following equation:*

$$f(x) = \mathcal{C}(\alpha,\beta)^{-1} \times x^{\alpha-1}e^{-\frac{x}{\beta}}$$

where $\mathcal{C}(\alpha,\beta) = \Gamma(\alpha)\beta^\alpha$ is the constant of integration. We will denote $X \sim G(\alpha,\beta)$.

In DYNARE this distribution may be specified as a prior in the `estimated_params` block (using the keyword `GAMMA_PDF`). The user has to specify the expectation and standard deviation of the distribution.

Proposition 1.1. *If $X \sim G(\alpha,\beta)$, then the expectation and variance of X are:*

$$\mu = \alpha\beta$$
$$\sigma^2 = \alpha\beta^2$$

Proof. By definition the expectation is given by:

$$\begin{aligned}\mu &= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty x^\alpha e^{-\frac{x}{\beta}}\mathrm{d}x \\ &= \mathcal{C}(\alpha,\beta)^{-1}\mathcal{C}(1+\alpha,\beta) \\ &= \frac{\Gamma(1+\alpha)\beta^{1+\alpha}}{\Gamma(\alpha)\beta^\alpha} \\ &= \frac{\alpha\Gamma(\alpha)\beta^{1+\alpha}}{\Gamma(\alpha)\beta^\alpha} \\ &= \alpha\beta\end{aligned}$$

By definition of the second order moment we have:

$$\begin{aligned}\mathbb{E}[X^2] &= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty x^{\alpha+1}e^{-\frac{x}{\beta}}\,\mathrm{d}x\\ &= \mathcal{C}(\alpha,\beta)^{-1}\mathcal{C}(2+\alpha,\beta)\\ &= \frac{\Gamma(2+\alpha)\beta^{2+\alpha}}{\Gamma(\alpha)\beta^{\alpha}}\\ &= \frac{(\alpha+1)\alpha\Gamma(\alpha)\beta^{2+\alpha}}{\Gamma(\alpha)\beta^{\alpha}}\\ &= \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2\end{aligned}$$

and

$$\sigma^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = \alpha\beta^2$$

□

DYNARE computes α and β from these moments, we have:

$$\begin{aligned}\alpha &= \frac{\mu^2}{\sigma^2}\\ \beta &= \frac{\sigma^2}{\mu}\end{aligned} \tag{1}$$

Proposition 1.2. *The mode of $X \sim G(\alpha,\beta)$ is:*

$$m = \begin{cases} 0 & \text{if } \alpha \leq 1,\\ (\alpha-1)\beta & \text{otherwise.}\end{cases}$$

Proof. We have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= \mathcal{C}(\alpha,\beta)^{-1}\times\left\{(\alpha-1)x^{\alpha-2}e^{-\frac{x}{\beta}} - \frac{1}{\beta}x^{\alpha-1}e^{-\frac{x}{\beta}}\right\}\\ &= \mathcal{C}(\alpha,\beta)^{-1}x^{\alpha-1}e^{-\frac{x}{\beta}}\times\left\{-\frac{1-\alpha}{x} - \frac{1}{\beta}\right\}\end{aligned}$$

If $\alpha \leq 1$ the density is monotone decreasing and we have a vertical asymptote at zero. Consequently, if $\alpha \leq 1$ we have $m = 0$. If $\alpha > 1$, ${}^{\mathrm{d}}/_{\mathrm{d}x}f(x)$ non negative if and only if:

$$-\frac{1-\alpha}{x} - \frac{1}{\beta} \geq 0$$

equivalently, we have:

$$x \leq (\alpha-1)\beta$$

If the shape parameter α is (strictly) greater than one, we have $m = (\alpha-1)\beta$. □

With an asymmetric distribution it may seem more sensible to define the parameters from the mode and the variance. We have:

$$\begin{cases} m &= (\alpha-1)\beta\\ \sigma^2 &= \alpha\beta^2\end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha &= 1 + \frac{m}{\beta}\\ 0 &= \beta^2 + m\beta - \sigma^2\end{cases}$$

One can easily check that the quadratic equation has always two distinct real solutions, one positive and one negative. We have:

$$\begin{aligned} \alpha &= 1 - \frac{2}{1 - \sqrt{1 + 4\left(\frac{\sigma}{m}\right)^2}} \\ \beta &= -\frac{m}{2}\left(1 - \sqrt{1 + 4\left(\frac{\sigma}{m}\right)^2}\right) \end{aligned} \tag{2}$$

1.1. **Gamma type 2 & 1 distributions.**

Definition 1.3. *Let $X > 0$ be a real random variable with a gamma distribution parametrized by a shape $\frac{\nu}{2} > 0$ and scale $\frac{2}{s} > 0$. We will denote $X \sim G_2(\nu, s) \equiv G\left(\frac{\nu}{2}, \frac{2}{s}\right)$ and say that X has a gamma-2 distribution.*

Definition 1.4. *Let $Y = \sqrt{X}$ with $X \sim G_2(\nu, s)$, we say that Y has a gamma-1 distribution and denote $Y \sim G_1(\nu, s)$.*

These distribution are not implemented in DYNARE, but can be easily built from the gamma distribution.

Proposition 1.3. *The densities of the gamma-2 and gamma-1 are respectively:*

$$f_X(x) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times x^{\frac{\nu}{2}-1} e^{-\frac{sx}{2}}$$

and

$$f_Y(y) = \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{\nu-1} e^{-\frac{s}{2}y^2}$$

where $\widetilde{\mathcal{C}}(\alpha, \beta) = \mathcal{C}(\alpha, \beta)/2$.

Proof. The density of the gamma-2 distribution is easily obtained from density of the gamma distribution. The first two moments of the gamma-2 distribution are directly obtained from proposition (1.1). The density of the gamma-1 distribution is given by:

$$f_Y(y) = f_X(h^{-1}(y)) \times \left|\frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y)\right|$$

where $h(x) \equiv \sqrt{x}$ and f_X denotes the density of the gamma-2 distribution:

$$f_X(x) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times x^{\frac{\nu}{2}-1} e^{-\frac{sx}{2}}$$

Substituting f_X and the h^{-1} into the definition of f_Y, we get:

$$\begin{aligned} f_Y(y) &= \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times y^{\nu-2} e^{-\frac{sy^2}{2}} \times \left|\frac{\mathrm{d}}{\mathrm{d}y} y^2\right| \\ \Leftrightarrow f_Y(y) &= 2\mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{\nu-2} e^{-\frac{sy^2}{2}} y \end{aligned}$$

so that:

$$f_Y(y) = \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{\nu-1} e^{-\frac{s}{2}y^2} \tag{3}$$

where $\widetilde{\mathcal{C}}(\alpha, \beta) = \mathcal{C}(\alpha, \beta)/2$ is the constant of integration. □

Proposition 1.4. *If $X \sim G_2(\nu, s)$, then the expectation and variance of X are:*

$$\mu = \frac{\nu}{s}$$
$$\sigma^2 = \frac{2\nu}{s^2}$$

Proof. By substituting the definition of the gamma-2 distribution in proposition 1.1. □

Proposition 1.5. *If $Y \sim G_1(\nu, s)$, then the expectation and variance of Y are:*

$$\mu = \sqrt{\frac{2}{s}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$
$$\sigma^2 = \frac{\nu}{s} - \mu^2$$

Proof. The expectation of this distribution is defined by:

$$\begin{aligned}
\mu &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty y^\nu e^{-\frac{s}{2}y^2} \mathrm{d}y \\
&= \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty x^{\frac{\nu}{2}-\frac{1}{2}} e^{-\frac{s}{2}x} \mathrm{d}x \\
&= \frac{\Gamma\left(\frac{\nu}{2}+\frac{1}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}+\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}}
\end{aligned}$$

So that

$$\mu = \sqrt{\frac{2}{s}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$

For the second order moment, we have:

$$\mathbb{E}\left[Y^2\right] = \mathbb{E}[X]$$

where $X \sim G_2(\nu, s)$. Using proposition 1.4 we obtain:

$$\mathbb{E}\left[Y^2\right] = \frac{\nu}{s}$$

□

1.2. **Chi-squared and Exponential distributions.** A number of distributions may be defined as special cases of the Gamma distribution. A chi-squared distribution with ν degrees of freedom, $\chi^2(\nu)$, is a gamma distribution: $G\left(\frac{\nu}{2}, 2\right)$. The chi-squared prior is not implemented in DYNARE but obviously the user can obtain it by carefully choosing the expectation and variance of the gamma prior (that is, by setting $\mu = \nu$ and $\sigma^2 = 2\nu$). As long as the variance is twice the expectation, the prior is a chi-squared distribution. An exponential distribution with expectation λ^{-1}, $\xi(\lambda)$, is also a gamma distribution: $G\left(1, \frac{1}{\lambda}\right)$. Again the exponential prior is not implemented in DYNARE but is obtained from the gamma distribution as long as the prior expectation is the squared root of the prior variance. As a consequence, by using the gamma prior and setting $\mu = \sigma$ the user chooses a distribution whose mode is zero.

1.3. **Shifted gamma distribution.** The support of the gamma distribution is usually the positive real line. In DYNARE the user has the possibility to shift the support of this distribution. This may be useful, for instance, if someone wants to estimate the elasticity of substitution of a CES production function with the (deterministic) belief that this elasticity has to be greater than one (Cobb-Douglas technology). The density is then defined with three parameters $\alpha > 0$ (shape), $\beta > 0$ (scale) and δ (location, the lower bound of the distribution's support):

$$f(x) = \mathcal{C}(\alpha,\beta)^{-1} \times (x-\delta)^{\alpha-1} e^{-\frac{x-\delta}{\beta}} \tag{4}$$

where the constant of integration is defined as before. Obviously this shift affects the first moment ($\mu = \delta + \alpha\beta$) and the mode of the distribution (the same shift applies) but not the variance.

2. Inverted gamma distribution

Definition 2.1. *Let X be a gamma distributed random variable with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. Then $Z = X^{-1}$ is said to be inverted gamma distributed, $Z \sim IG(\alpha,\beta)$.*

Proposition 2.1. *The density of the continuous random variable $Z \sim IG(\alpha,\beta)$ is:*

$$f(z) = \mathcal{C}(\alpha,\beta)^{-1} \times z^{-\alpha-1} e^{-\frac{1}{\beta z}}$$

where $\mathcal{C}(\alpha,\beta) = \Gamma(\alpha)\beta^{\alpha}$ is the constant of integration.

Proof. Let f_X denote the density of the gamma distribution and define $h(x) = \frac{1}{x}$. The density of the inverted gamma distribution is defined as follows:

$$\begin{aligned} f_Z(z) &= f_X(h^{-1}(z)) \times \left| \frac{\mathrm{d}}{\mathrm{d}z} h^{-1}(z) \right| \\ &= \mathcal{C}(\alpha,\beta)^{-1} \times z^{-\alpha+1} e^{-\frac{1}{\beta z}} \times \left| \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{z} \right| \\ &= \mathcal{C}(\alpha,\beta)^{-1} \times z^{-\alpha-1} e^{-\frac{1}{\beta z}} \end{aligned}$$

□

Proposition 2.2. *The expectation and variance of $Z \sim IG(\alpha,\beta)$ are:*

$$\begin{aligned} \mu &= \frac{1}{\beta(\alpha-1)} \\ \sigma^2 &= \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)} \qquad \textit{for any } \alpha \geq 2 \end{aligned}$$

Proof. By definition of the Inverse Gamma pdf, we have:

$$\begin{aligned}
\mu &= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty z^{-\alpha}e^{-\frac{1}{\beta z}}\mathrm{d}z \\
&= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty u^{\alpha-2}e^{-\frac{u}{\beta}}\mathrm{d}u \\
&= \frac{\mathcal{C}(\alpha-1,\beta)}{\mathcal{C}(\alpha,\beta)} \\
&= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \\
&= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{(\alpha-1)\Gamma(\alpha-1)\beta^{\alpha}} \\
&= \frac{1}{(\alpha-1)\beta}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[Z^2] &= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty z^{-\alpha+1}e^{-\frac{1}{\beta z}}\mathrm{d}z \\
&= \mathcal{C}(\alpha,\beta)^{-1}\int_0^\infty u^{\alpha-3}e^{-\frac{u}{\beta}}\mathrm{d}u \\
&= \frac{\mathcal{C}(\alpha-2,\beta)}{\mathcal{C}(\alpha,\beta)} \\
&= \frac{\Gamma(\alpha-2)\beta^{\alpha-2}}{\Gamma(\alpha)\beta^{\alpha}} \\
&= \frac{\Gamma(\alpha-2)\beta^{\alpha-2}}{(\alpha-2)(\alpha-1)\Gamma(\alpha-2)\beta^{\alpha}} \\
&= \frac{1}{(\alpha-1)(\alpha-2)\beta^2}
\end{aligned}$$

so that

$$\begin{aligned}
\sigma^2 &= \frac{1}{(\alpha-1)(\alpha-2)\beta} - \frac{1}{(\alpha-1)^2\beta^2} \\
&= \frac{(\alpha-1)}{(\alpha-1)^2\beta^2(\alpha-2)} - \frac{\alpha-2}{(\alpha-1)^2\beta^2(\alpha-2)} \\
&= \frac{1}{(\alpha-1)^2\beta^2(\alpha-2)}
\end{aligned}$$

□

The system of equations defining the expectation and the variance can be solved for the shape and scale parameters:

$$\begin{aligned}
\alpha &= 2 + \left(\frac{\mu}{\sigma}\right)^2 \\
\beta &= \frac{1}{\mu\left[1+\left(\frac{\mu}{\sigma}\right)^2\right]}
\end{aligned} \tag{5}$$

This distribution is not implemented in DYNARE.

Proposition 2.3. *The mode of $Z \sim IG(\alpha, \beta)$ is strictly positive:*

$$m = \frac{1}{\beta(1+\alpha)}$$

Proof. We have:

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = \mathcal{C}(\alpha, \beta)^{-1} z^{-\alpha-2} e^{-\frac{1}{\beta z}} \left\{ \frac{1}{\beta z} - (1+\alpha) \right\}$$

There exists only one value of z such that this first derivate is zero:

$$\frac{\mathrm{d}}{\mathrm{d}z} f(z) = 0 \Leftrightarrow z = \frac{1}{\beta(1+\alpha)}$$

□

Again one may prefer to define this distribution by specifying the mode and the variance (see below for the inverse gamma-2 distribution). Note that it is also possible to define this distribution by specifying the expectation and the mode. We have:

$$\begin{cases} \mu &= \frac{1}{\beta(\alpha-1)} \\ m &= \frac{1}{\beta(\alpha+1)} \end{cases}$$

or equivalently:

$$\begin{cases} \mu &= \frac{1}{\beta(\alpha-1)} \\ m &= \mu\frac{\alpha-1}{(\alpha+1)} \end{cases}$$

We can solve the second equation for α. We have:

$$\alpha = \left(1 - \frac{m}{\mu}\right)^{-1} \left(\frac{m}{\mu} + 1\right) = \frac{\mu + m}{\mu - m}$$

and by substitution:

$$\beta = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{\mu}\right) = \frac{\mu - m}{2\mu m}$$

2.1. **Inverted gamma-2 and gamma-1 distributions.**

Definition 2.2. *Let $X > 0$ be a real random variable with a gamma-2 distribution, $X \sim G\left(\frac{\nu}{2}, \frac{2}{s}\right)$. $Y = X^{-1}$ is said to have an inverted gamma-2 distribution $Y \sim IG_2(\nu, s)$.*

Proposition 2.4. *The probability density function of $Y \sim IG_2(\nu, s)$ is:*

$$f_Y(y) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \times y^{-\frac{\nu}{2}-1} e^{-\frac{s}{2y}}$$

Proof. Direct from proposition 2.1 with $\alpha = {}^{\nu}/_{2}$ and $\beta = {}^{2}/_{s}$. □

Proposition 2.5. *The expectation and variance of $Y \sim IG_2(\nu, s)$ are:*

$$\mu = \frac{s}{\nu - 2}$$
$$\sigma^2 = \frac{2\mu^2}{\nu - 4}$$

Proof. Direct from proposition 2.2 with $\alpha = {}^{\nu}/_{2}$ and $\beta = {}^{2}/_{s}$. □

The inverted gamma-2 distribution is implemented in DYNARE as a prior, using the keyword `INV_GAMMA2_PDF`. The user has to specify μ and σ, and DYNARE solves the two equations given in proposition 2.5 for the scale and shape parameters:

$$
\begin{aligned}
s &= 2\mu\left(1+\frac{\mu^2}{\sigma^2}\right) \\
\nu &= 2\left(2+\frac{\mu^2}{\sigma^2}\right)
\end{aligned} \tag{6}
$$

This distribution is often used as a prior for the variance of a structural shock or measurement error. Note that the sole difference between an inverted gamma distribution and the inverted gamma-2 distribution is in the parametrization of the shape and scale parameters. If the prior distribution is defined by its first and second moments, this difference does not matter.

Proposition 2.6. *The mode of $Y \sim IG_2(\nu, s)$ is:*

$$m = \frac{s}{\nu+2} > 0$$

Proof. We have:

$$\frac{\mathrm{d}}{\mathrm{d}y} f_Y(y) = \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\frac{\nu}{2}-2} e^{-\frac{s}{2y}} \left\{\frac{s}{2y} - \left(\frac{\nu}{2}+1\right)\right\}$$

There exists only one value such that the first derivate is zero:

$$\frac{\mathrm{d}}{\mathrm{d}y} f_Y(y) = 0 \Leftrightarrow y = \frac{s}{\nu+2}$$

□

One can define the prior using the mode and the variance (or alternatively the mode and the mean). Substituting the expression for the mode in the expression for σ^2 and rearranging we obtain the following cubic equation for ν:

$$\varphi_\delta(\nu) \equiv \nu^3 - (8+\delta)\nu^2 + (20-4\delta)\,\nu - 16 - 4\delta = 0 \tag{7}$$

where $\delta = 2\left({}^{m}/_{\sigma}\right)^2$.

Proposition 2.7. *The cubic equation 7 has only one real solution greater than four.*

Let $\nu^\star(\delta)$ be the pertinent root of φ. We then have $s^\star = m(\nu^\star(\delta)+2)$ by inverting the mode formula given in proposition 2.6.

Proof. Let $\bar{\nu}_1(\delta)$ and $\bar{\nu}_2(\delta)$ be the roots of the second order polynomial $\varphi'_\delta(\nu)$. One can show that they are given by :

$$\bar{\nu}_1(\delta) = \frac{16+2\delta-2\sqrt{\delta^2+28\delta+4}}{6} \leq 2 \quad \forall \delta \geq 0$$

and

$$\bar{\nu}_2(\delta) = \frac{16+2\delta+2\sqrt{\delta^2+28\delta+4}}{6} \geq \frac{10}{3} \quad \forall \delta \geq 0$$

Let $\tilde{\nu}(\delta) = \frac{8}{3} + \frac{\delta}{3}$ be the root of $\varphi''_\delta(\nu)$, we have :

$$\bar{\nu}_1(\delta) < \tilde{\nu}(\delta) < \bar{\nu}_2(\delta)$$

for any value of δ. As a consequence, we have $\varphi'_\delta(\tilde{\nu}(\delta)) < 0$ and also $\varphi_\delta(\bar{\nu}_2(\delta)) < 0$. Knowing that $\varphi_\delta(\nu)$ is monotone increasing in $[\bar{\nu}_2(\delta), +\infty)$ and $\varphi_\delta(4) = -36\delta \leq 0$, the biggest root of the third order polynomial has to be greater than four.

□

In practice we instead usually define the priors over standard deviations, that is over the square root of the variance. This motivates the following definition.

Definition 2.3. *Let $X > 0$ be a real random variable with a gamma-1 distribution, $X \sim G_1(\nu, s)$. $Y = X^{-1}$ is said to have an inverted gamma-1 distribution $Y \sim IG_1(\nu, s)$.*

Proposition 2.8. *The probability density function of $Y \sim IG_1(\nu, s)$ is:*

$$f_Y(y) = \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu-1} e^{-\frac{s}{2y^2}}$$

Proof. Let f_X denote the density of the gamma-1 distribution and define $h(x) = 1/x$. The density of the inverted gamma-1 distribution is defined as follows:

$$\begin{aligned} f_Y(y) &= f_X(h^{-1}(y)) \times \left|\frac{\mathrm{d}}{\mathrm{d}y} h^{-1}(y)\right| \\ &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu+1} e^{-\frac{s}{2y^2}} \frac{1}{y^2} \\ &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} y^{-\nu-1} e^{-\frac{s}{2y^2}} \end{aligned}$$

□

Proposition 2.9. *The expectation and variance of $Y \sim IG_1(\nu, s)$ are:*

$$\begin{aligned} \mu &= \sqrt{\frac{s}{2}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \\ \sigma^2 &= \frac{s}{\nu-2} - \mu^2 \end{aligned}$$

Proof. The first order moment is defined by:

$$\begin{aligned} \mu &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty y y^{-\nu-1} e^{-\frac{s}{2y^2}} \mathrm{d}y \\ &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \frac{1}{2} \int_0^\infty x^{\frac{\nu}{2}-\frac{1}{2}-1} e^{-\frac{s}{2}x} \mathrm{d}x \\ &= \mathcal{C}\left(\frac{\nu}{2}, \frac{2}{s}\right)^{-1} \int_0^\infty x^{\frac{\nu-1}{2}-1} e^{-\frac{s}{2}x} \mathrm{d}x \\ &= \frac{\Gamma\left(\frac{\nu-1}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu-1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}} \\ &= \sqrt{\frac{s}{2}} \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \end{aligned}$$

The second order un-centered moment is defined by:

$$\begin{aligned}
\mathbb{E}[Y^2] &= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty y^2 y^{-\nu-1}e^{-\frac{s}{2y^2}}\mathrm{d}y\\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\int_0^\infty y^{-\nu+1}e^{-\frac{s}{2y^2}}\mathrm{d}y\\
&= \widetilde{\mathcal{C}}\left(\frac{\nu}{2},\frac{2}{s}\right)^{-1}\frac{1}{2}\int_0^\infty x^{\frac{\nu}{2}-2}e^{-\frac{s}{2}x}\mathrm{d}x\\
&= \frac{\Gamma\left(\frac{\nu}{2}-1\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}-1}}{\Gamma\left(\frac{\nu}{2}\right)\left(\frac{2}{s}\right)^{\frac{\nu}{2}}}\\
&= \frac{s}{2}\frac{\Gamma\left(\frac{\nu-2}{2}\right)}{\frac{\nu-2}{2}\Gamma\left(\frac{\nu-2}{2}\right)}\\
&= \frac{s}{\nu-2}
\end{aligned}$$

So that $\sigma^2 = \frac{s}{\nu-2} - \mu^2$. □

The inverted gamma-1 distribution is implemented in DYNARE as a prior (using the keywords `INV_GAMMA1_PDF` or `INV_GAMMA_PDF`). The user has to specify μ and σ, and DYNARE solves the equations given in proposition 2.9 for the scale and shape parameters. There is no closed form solution in this case, a numerical approach is used. DYNARE first solves for ν in the following equation[1]:

$$2\Gamma\left(\frac{\nu}{2}\right)^2\mu^2 = (\sigma^2+\mu^2)(\nu-2)\Gamma\left(\frac{\nu-1}{2}\right)^2$$

and then computes:

$$s = (\sigma^2+\mu^2)(\nu-2)$$

This is done in the `m` file `inverse_gamma_specification.m`. Note that in the case of an infinite variance we have a closed form solution:

$$\begin{aligned}
\nu &= 2\\
s &= \frac{2}{\pi}\mu^2
\end{aligned}$$

Proposition 2.10. *The mode of* $Y \sim IG_1(\nu, s)$ *is:*

$$m = \sqrt{\frac{\nu-1}{s}}$$

Proof. We have:

$$\frac{\mathrm{d}}{\mathrm{d}y}f_Y(y) \propto x^{\nu}e^{-\frac{s}{2}x^2}\left\{\frac{\nu-1}{x^2} - s\right\}$$

which is zero iff $x = m$. □

One can define the prior using the mode and the variance. Substituting the expression for the mode in the expression for σ^2 and rearranging we obtain:

$$\sigma^2 m^2 = \frac{\nu-1}{\nu-2} - \frac{\nu-1}{2}\left(\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right)^2$$

[1] Because Γ

Dynare numerically solves this equation for ν and computes $s = {}^{\nu-1}/_{m^2}$. Note that in the case of an infinite variance we have a closed form solution:

$$\nu = 2$$
$$s = \frac{1}{m^2}$$

☞ The inverse gamma-1(-2) is usually used as a prior for the standard deviation (resp. variance) of a structural (or measurement) shock. This is because in linear models with gaussian perturbation, the Normal (for the parameters) – Inverse Gamma (for the variance of the error) prior is conjugate. Obviously this is not true for DSGE models, there is no computational advantage in choosing the inverse gamma prior.

3. Beta distributions

Definition 3.1. *The Beta function is defined as follows:*

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathrm{d}x$$

with $(\alpha,\beta) \in \mathbb{R}^2$.

Proposition 3.1. *The Beta function can be written using the Gamma function as follows:*

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Proposition 3.2. *Let* $X \sim G(\alpha,1)$ *and* $Y \sim G(\beta,1)$ *be two independent random variables. The random variable* $Z = {}^{X}/_{X+Y}$ *is said to be beta distributed,* $Z \sim \mathcal{B}(\alpha,\beta)$, *and its probability density function is given by:*

$$f_Z(z) = B(\alpha,\beta)z^{\alpha-1}(1-z)^{\beta-1}$$

Proposition 3.3. *The expectation and variance of* $Z \sim \mathcal{B}(\alpha,\beta)$ *are:*

$$\mu = \frac{\alpha}{\alpha+\beta}$$
$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

The system of equations defining the expectation and the variance can be solved for the shape and scale parameters:

$$\begin{aligned} \alpha &= \frac{(1-\mu)\mu^2}{\sigma^2} - \mu \\ \beta &= \left(\frac{(1-\mu)\mu}{\sigma^2} - 1\right)(1-\mu) \end{aligned} \tag{8}$$

The beta distribution can have one, two (the density is U-shaped if and only if $\alpha < 1$ and $\beta < 1$) or an infinity of modes (if $\alpha = \beta = 1$ the beta distribution collapses in a uniform distribution). The following proposition gives the mode of the beta in the unimodal case.

Proposition 3.4. *The mode of* $Z \sim \mathcal{B}(\alpha, \beta)$ *is:*

$$m = \begin{cases} 0, & \textit{if } \alpha < 1 \textit{ and } \beta \geq 1 \textbf{\textit{ or }} \alpha = 1 \textit{ and } \beta > 1 \\ 1, & \textit{if } \alpha = 1 \textit{ and } \beta < 1 \textbf{\textit{ or }} \alpha > 1 \textit{ and } \beta \leq 1 \\ \frac{\alpha-1}{\alpha+\beta-2} & \textit{if } \alpha \geq 1 \textit{ and } \beta \geq 1 \end{cases}$$

In the first two cases (the mode is on the right or the left of the distribution support), depending on the values of α and β the density probability function can be strictly convex (vertical asymptote at 0 or 1), strictly concave or linear. Again, considering the interior mode case, one can define the prior from the mode and the variance.

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